Dynamics

Introduction to Robotics: Discussion 8

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ADMINISTRIVA

- Midterm I Results: Oct 15th
- HW6 Self Grades: Oct 22nd
- HW7 Due: Oct 22nd
- HW8 Released: Oct 22nd

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TERMINOLOGY

Forward Kinematics

– Given joint positions, find end effector coordinates

Inverse Kinematics

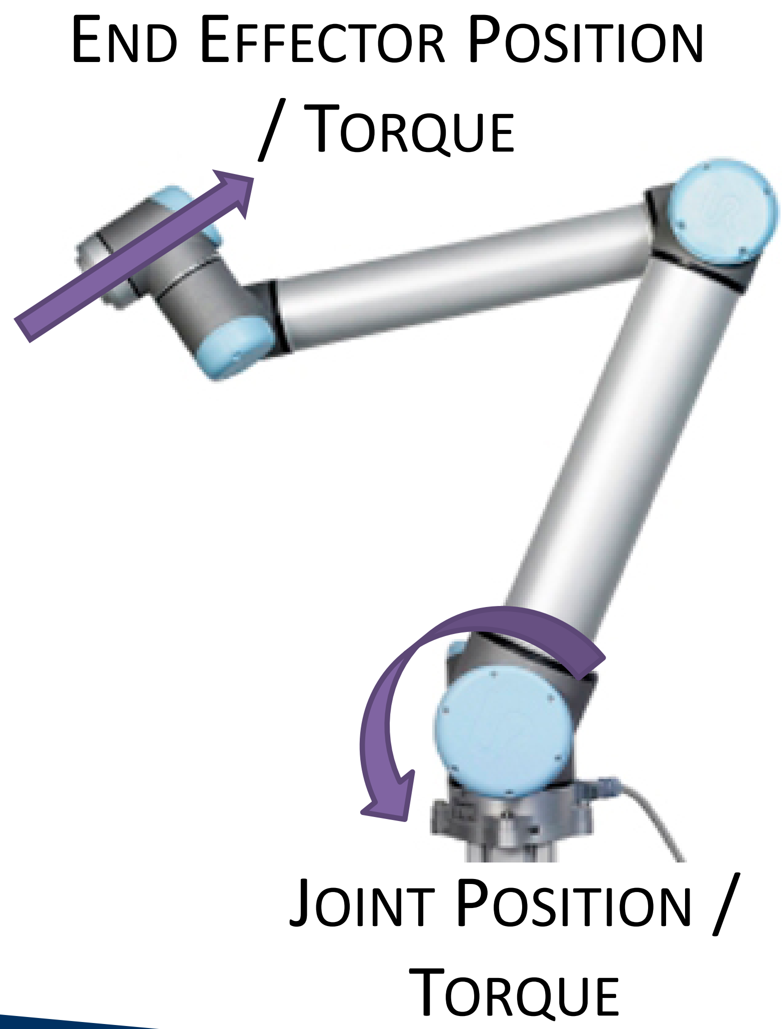
– Given end effector coordinates, find required joint positions

Forward Dynamics

– Given joint torques, find end effector forces/torques

Inverse Dynamics

– Given a desired end effector force/torque, find required joint torques



TWISTS & WRENCHES

Linear and rotational velocities can be combined to form a twist

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} \in \mathbb{R}^{6\times 1}$$

$\boldsymbol{v} \in \mathbb{R}^{3\times 1}$ linear component

$\boldsymbol{\omega} \in \mathbb{R}^{3\times 1}$ rotational component

Linear and rotational forces can be combined to form a wrench

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{\tau} \end{bmatrix} \in \mathbb{R}^{6\times 1}$$

$\boldsymbol{f} \in \mathbb{R}^{3\times 1}$ linear component

$\boldsymbol{\tau} \in \mathbb{R}^{3\times 1}$ rotational component

Full use of $SE(3)$ for dynamics in E106B

NEWTON/EULER DYNAMICS

Newton's 2nd Law:

$$f = \frac{d}{dt}(mv)$$ Linear Force = change in linear momentum

Euler's Equation:

$$\tau = \frac{d}{dt}(I'\omega^s)$$ Rotational Torque = change in angular momentum

$$I' = RIR^T$$ Inertia w.r.t. inertial frame

NEWTON/EULER DYNAMICS

Assume 2D, no change in mass or inertia w.r.t. time:

$$f = \frac{d}{dt}(mv)$$

$$\tau = \frac{d}{dt}(I'\omega^s) \qquad I' = RIR^T$$

$$\tau = RIR^T\dot{\omega}^s + \dot{R}IR^T\omega^s + RI\dot{R}^T\omega^s$$

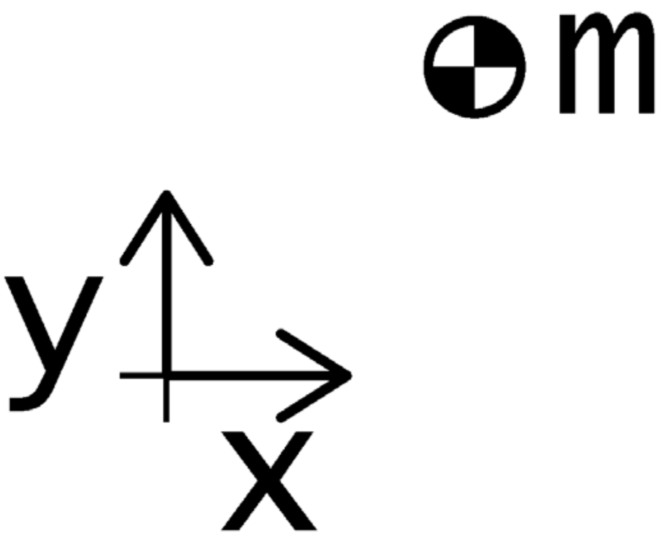
$$\tau = I'\dot{\omega}^s + \omega^s \times I'\omega^s - I'\omega^s \times \omega^s$$

$$f = ma$$

$$\tau = I'\ddot{\theta}^s$$

EXAMPLE I: PARTICLE

$$f = ma$$
$$\tau = I'\ddot{\theta}$$

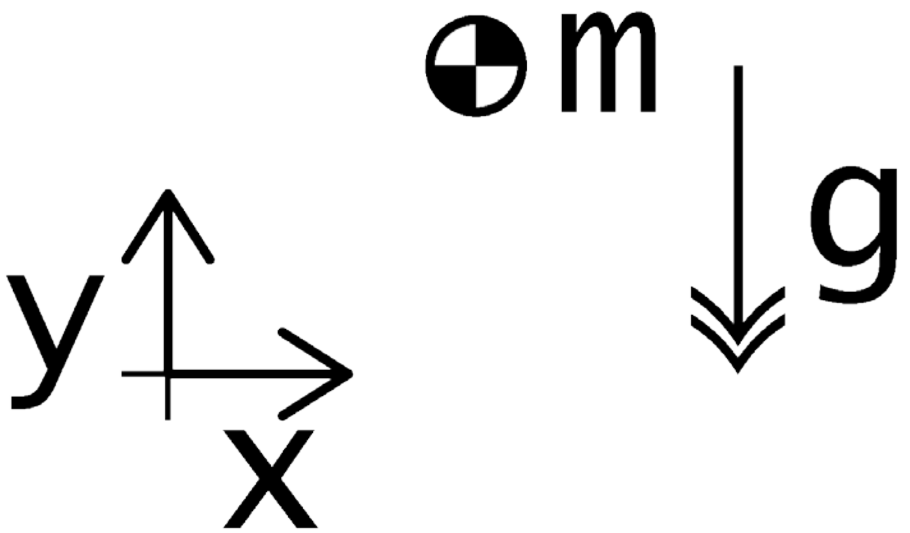


$$f_X = m\ddot{x} = 0$$
$$\rightarrow \ddot{x} = 0$$

$$f_Y = m\ddot{y} = 0$$
$$\rightarrow \ddot{y} = 0$$

EXAMPLE II: PARTICLE & GRAVITY

$$f = ma$$
$$\tau = I'\ddot{\theta}$$



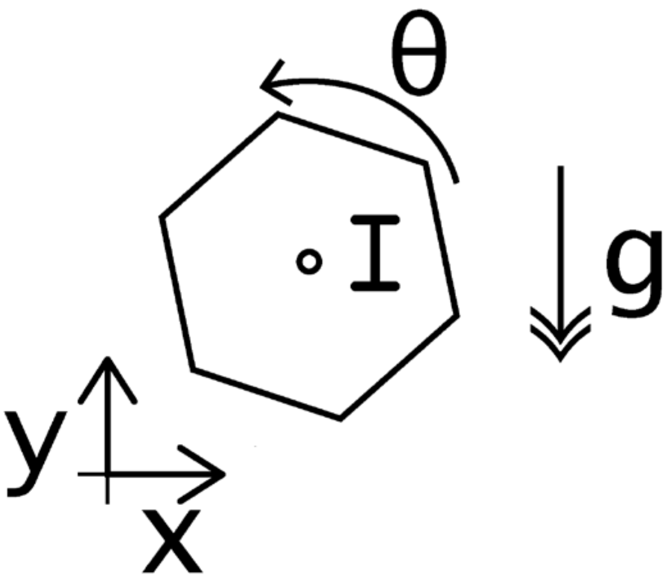
$$f_X = m\ddot{x} = 0$$
$$\rightarrow \ddot{x} = 0$$

$$f_Y = m\ddot{y} = \boldsymbol{-mg}$$
$$\rightarrow \ddot{y} = -g$$

Gravitational force $= \boldsymbol{-mg}$

EXAMPLE III: BODY & GRAVITY

$$f = ma$$
$$\tau = I'\ddot{\theta}$$



$$f_X = m\ddot{x} = 0$$
$$\rightarrow \ddot{x} = 0$$

$$f_Y = m\ddot{y} = -mg$$
$$\rightarrow \ddot{y} = -mg$$

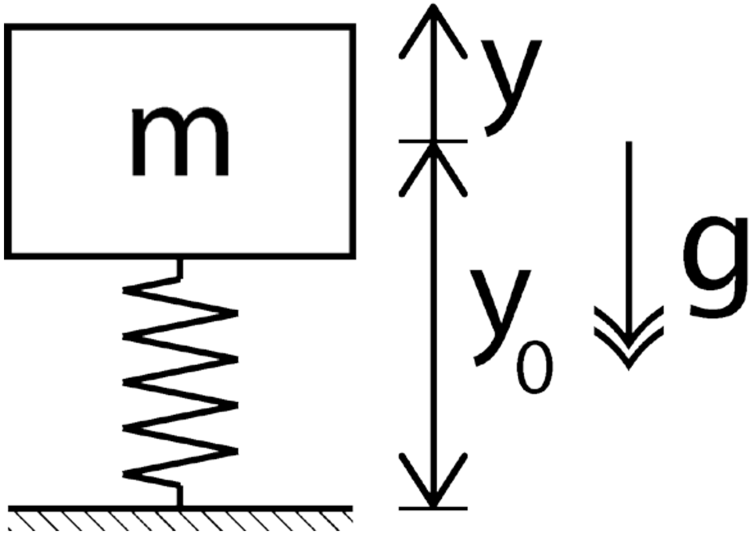
$$\tau = I'\ddot{\theta} = 0$$
$$\rightarrow \ddot{\theta} = 0$$

EXAMPLE IV: SPRING MASS

$$f = ma$$
$$\tau = I'\ddot{\theta}$$

A **linear spring** can be characterized by the equation:

$$\boldsymbol{f_Y = -k(y - y_0)}$$

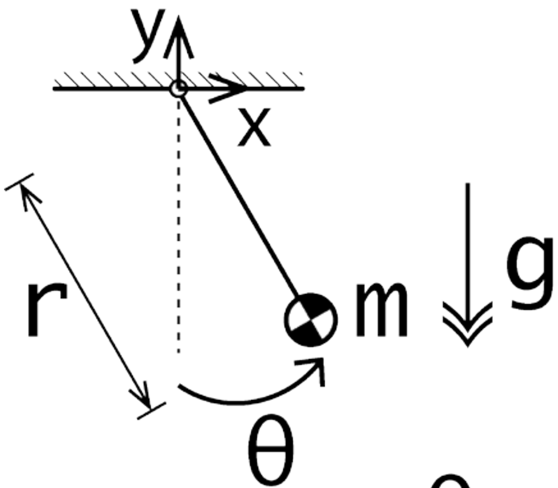


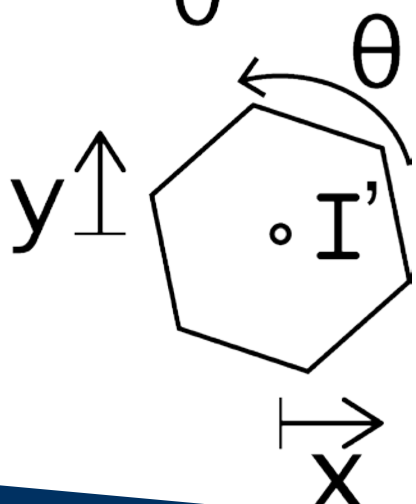
$$f_Y = m\ddot{y} = -k(y - y_0) - mg$$

$$\rightarrow \ddot{y} = -\frac{k}{m}(y - y_0) - g$$

$$\rightarrow y_0 = \frac{gm}{k} \qquad \rightarrow \ddot{y} = -\frac{k}{m}y$$

EXAMPLE IV: PENDULUM



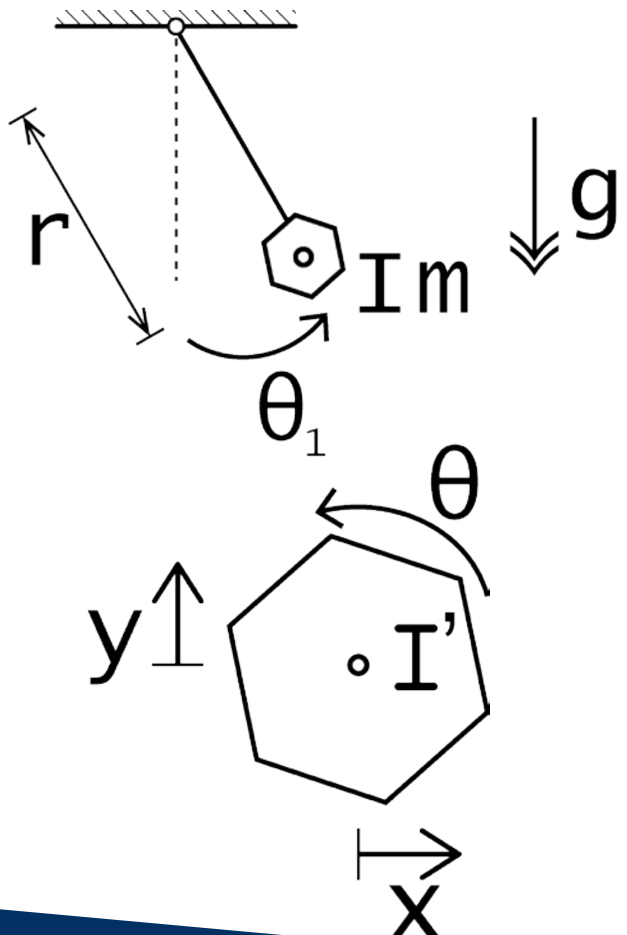


$$f = ma$$
$$\tau = I'\ddot{\theta}$$

We can rewrite this system as a rotating object with the inertia: $\boldsymbol{I' = mr^2}$

$$\tau = I'\ddot{\theta} = -rmgs$$
$$mr^2\ddot{\theta} = -rmgs$$
$$\ddot{\theta} = -\frac{g}{r}s$$

EXAMPLE V: PENDULAR ROD



$$f = ma$$
$$\tau = I'\ddot{\theta}$$

We can rewrite this system as a rotating object with the inertia:

$$\boldsymbol{I' = mr^2 + I}$$

This is known as the parallel axis theorem

$$\tau = I'\ddot{\theta} = -rmgs$$
$$(mr^2 + I)\ddot{\theta} = -rmgs$$
$$\ddot{\theta} = -\frac{rmg}{mr^2 + I}s$$

LAGRANGIAN DYNAMICS

Lagrangian = Kinetic Energy - Potential Energy

$$L = T - V$$

The change in the Lagrange is equal to the non-conservative forces:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

where q_i is a generalized coordinate

where γ_i is the corresponding non-conservative force

EXAMPLE I: PARTICLE

$$L = T - V$$

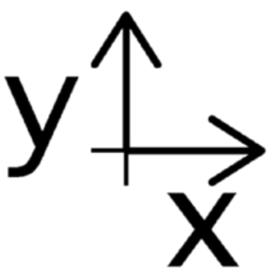
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

Kinetic Energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

Potential Energy $V = 0$

Lagrangian $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} = 0 \qquad \rightarrow \ddot{x} = 0$$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} = 0 \qquad \rightarrow \ddot{y} = 0$$

EXAMPLE II: PARTICLE & GRAVITY

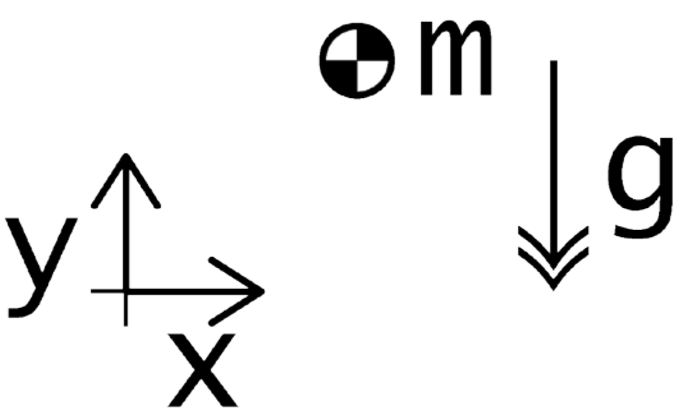
$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

Kinetic Energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

Potential Energy $V = mgy$

Lagrangian $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} = 0 \qquad \rightarrow \ddot{x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + mg = 0 \qquad \rightarrow \ddot{y} = -g$$

EXAMPLE III: BODY & GRAVITY

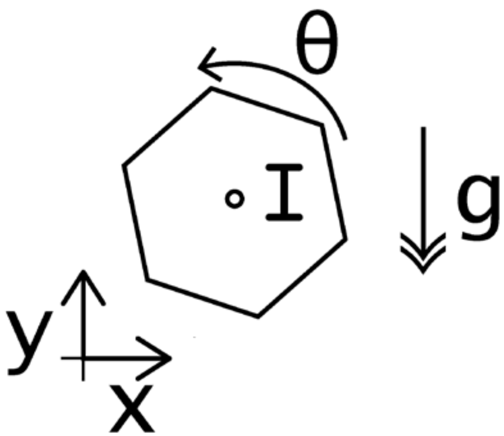
$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

Kinetic Energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2$

Potential Energy $V = mgy$

Lagrangian $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 - mgy$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} = 0 \quad \rightarrow \ddot{x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + mg = 0 \quad \rightarrow \ddot{y} = -g$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = I\ddot{\theta} = 0 \quad \rightarrow \ddot{\theta} = 0$$

EXAMPLE IV: SPRING MASS

$$L = T - V$$

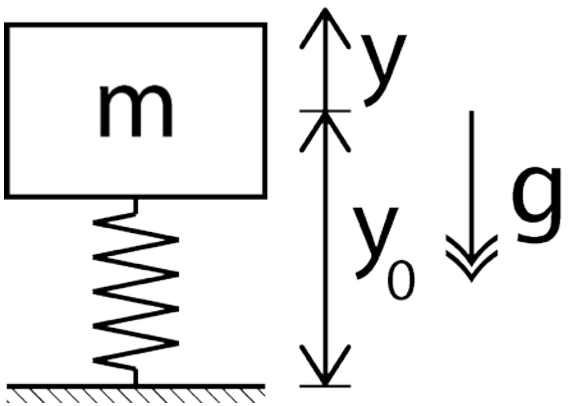
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

Kinetic Energy $T = \frac{1}{2}m(\dot{y}^2)$

Potential Energy $V = mgy + \mathbf{\frac{1}{2}k(y - y_0)^2}$

potential energy stored in a spring

Lagrangian $L = \frac{1}{2}m(\dot{y}^2) - \frac{1}{2}k(y - y_0)^2 - mgy$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y} \qquad \frac{\partial L}{\partial y} = -k(y - y_0) - mg$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + k(y - y_0) + mg = 0$$

$$\ddot{y} = -\frac{k}{m}(y - y_0) - g \qquad y_0 = \frac{gm}{k} \rightarrow \ddot{y} = -\frac{k}{m}y$$

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EXAMPLE V: PENDULUM

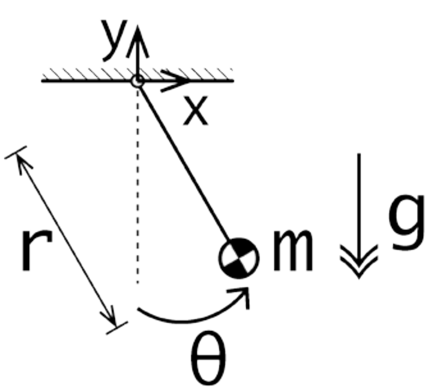
$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

Kinetic Energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(r^2c^2\dot{\theta}^2 + r^2s^2\dot{\theta}^2\right) = \frac{1}{2}mr^2\dot{\theta}^2$

Potential Energy $V = mgy = -mgrc$

Lagrangian $L = \frac{1}{2}mr^2\dot{\theta}^2 + mgrc$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgrs$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = mr^2\ddot{\theta} + mgrs$$

$$\ddot{\theta} = -\frac{g}{r}s$$

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